Components, Coalgebras, and Chu Spaces

Jean-Bernard Stefani

INRIA Grenoble-Rhône-Alpes

Outline

1. Motivations
2. Components as coalgebras
   - Coalgebras
   - Components as coalgebras
3. Component and Chu Spaces

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1 Motivations

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- Components as coalgebras

3 Component and Chu Spaces

Two questions re Software Components in PiCoq

1. Semantics ?
2. Calculus ?

Semantics

- Formal component metamodel
- Various form of behavioral composition
- Sharing & aspects
- Extensions : stochastic & hybrid models

Calculus

- Process calculus for component constructs
- Universality wrt component metamodel
  - *Any r.e. component can be realized*
- Coalgebraic semantics

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Circular phenomena in computer science

- Streams
- Infinite data structures
- Labelled transition systems
- Self-applicative & reflective programs

Algebras

- $x : X \times X \to X$
  $y : X \times X \times X \to X$
  $z : 1 \to X$
- $f : \Sigma(X) \to X$
- homomorphism: from $f : \Sigma(A) \to A$ to $g : \Sigma(B) \to B$
  $h : A \to B$ extends to $\Sigma(h) : \Sigma(A) \to \Sigma(B)$
- initial Σ-algebra: $id_{\Sigma_*} : \Sigma(\Sigma_*) \to \Sigma_*$
  there exists a unique homomorphism
  h_f from $id_{\Sigma_*} : \Sigma(\Sigma_*) \to \Sigma_*$ to $f : \Sigma(X) \to X$
- Σ_* is the least fixed point of operator Σ
  (Σ conceived as operator on sets)

Coalgebras

- $f : X \to \Gamma(X)$
- homomorphism: from $f : A \to \Gamma(A)$ to $g : B \to \Gamma(B)$
  $h : A \to B$ extends to $\Gamma(h) : \Gamma(A) \to \Gamma(B)$
- final Γ-coalgebra (Γ monotone): $id_{\Gamma^*} : \Gamma^* \to \Gamma(\Gamma^*)$
  there exists a unique homomorphism
  h_f from $f : A \to \Gamma(A)$ to $id_{\Gamma^*} : \Gamma^* \to \Gamma(\Gamma^*)$
- Γ^* is the greatest fixed point of operator Γ

Hypersets

- Simple set-based models: "concrete" and "intuitive"
- Working up-to bisimulation equivalence factored-in
- Greatest non-trivial fixed-points for key operators

What's an hyperset ?

- A non-well-founded set
  - $\Omega = \{\Omega\}$
  - $s = \langle a, s \rangle$
  - $x = \{\langle a, y \rangle, \langle b, x \rangle\}$
    $y = \{\langle c, z \rangle, \langle a, x \rangle\}$
    $z = \{\langle c, z \rangle, \langle b, x \rangle\}$
- The unique solution of a flat system of equation (Anti-Foundation Axiom)
  - A flat system of equation is a function: $e : X \to \mathcal{P}(X \cup A)$
    X a set of variables
    A a set disjoint from X
- A pointed coalgebra of the functor $\mathcal{P}(X \cup A)$

Fixed points of functors

- Functor: $\Gamma : \mathcal{U}_{afa} \to \mathcal{U}_{afa}$
  $\mathcal{U}_{afa}$ class of all (hyper)sets
- Monotone functor: $a \subseteq b \implies \Gamma(a) \subseteq \Gamma(b)$
- G fixed point of Γ is $G = \Gamma(G)$
- Least fixed point Γ_*: for any G fixed point $\Gamma_* \subseteq G$
  - Induction principle: to prove $\Gamma_* \subseteq G$, show that $\Gamma(G \cap \Gamma_*) \subseteq G$
- Greatest fixed point Γ^*: for any G fixed point $G \subseteq \Gamma^*$
  - Coinduction principle: to prove $G \subseteq \Gamma^*$, show that $G \subseteq \Gamma(G \cup \Gamma^*)$
- Every monotone functor Γ has a least and a greatest fixed point

Fixed points of functors

$\Gamma(a)$	Γ_*	Γ^* (with *FA*)	Γ^* (with *AFA*)
a	$\emptyset$	$\mathcal{U}_{wf}$	$\mathcal{U}_{afa}$
$\mathcal{P}(a)$	$\mathcal{U}_{wf}$	$\mathcal{U}_{wf}$	$\mathcal{U}_{afa}$
$A \times a$	$\emptyset$	$\emptyset$	infinite streams over A
$A \times a \times a$	$\emptyset$	$\emptyset$	infinite binary trees over A
$\mathcal{P}(A \times a)$	$\emptyset$	$\emptyset$	canonical LTS over A

Coalgebras

Theorem (Representation theorem for greatest fixed points)

If Γ is uniform (i.e. monotone + some other conditions)

$$\Gamma^* = \{solution(\mathcal{E}) \mid \mathcal{E} \text{ a } \Gamma\text{-coalgebra}\}$$

Definition

Let Γ be a monotone functor. A Γ-invariant is a predicate P on Γ^*, i.e. a subset of Γ^*, such that if $P(c)$ holds then $P(k)$ holds for all $k \in TC(c) \cap \Gamma^*$.

Theorem

Let Γ be a uniform functor, and let P be a predicate on Γ^*. Let P^* be the greatest Γ-invariant contained in P. Then $\langle P^*, id_{P^*} \rangle$ is a Γ-coalgebra.

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Fractal components as pointed coalgebras

Definition (The Fractal functor)

$\Gamma(X) = \mathcal{M}_f(X) \times \mathcal{P}(\mathcal{M}_f(\Sigma(X)) \times \mathcal{M}_f(\Sigma(X)) \times \mathcal{M}_f(X))$

- $\Sigma(X) = \coprod_{k \in \mathbb{N}} (\mathsf{L} \times \Delta(X))^k$
- $\Delta(X) = \mathsf{L} + \mathsf{V} + X$
- L set of names
- V set of values
- $\mathcal{M}_f(S)$ finite multisets of elements from S
- $\mathcal{P}(S)$ subsets of S

- $\Gamma^* = \Gamma(\Gamma^*)$ set of components
- $\Sigma(\Gamma^*)$ set of signals
- $\Delta(\Gamma^*)$ set of arguments

Fractal components as pointed coalgebras

- Fractal component are solutions to pointed equations: $e_x : X \to \Gamma(X)$
- Example:
  $x = \{y_1, y_2\} \cdot \{\langle\{s_1, s_2\}, \{r\}, y\rangle, \langle\{\mathtt{ping}\}, \{\mathtt{ok}\}, x\rangle\}$
  $y = \{y_1\} \cdot \{\langle\{s_3\}, \{\mathtt{ok}\}, x\rangle\}$
  $s_1 = \langle \mathtt{op} : \mathtt{lock}\rangle$
  $s_2 = \langle \mathtt{op} : \mathtt{get}, \mathtt{arg} : \mathtt{snd}\rangle$
  $r = \langle \mathtt{return} : y_2\rangle$
  $s_3 = \langle \mathtt{op} : \mathtt{insert}, \mathtt{arg} : y_2\rangle$
- in component $x = \{y_1, y_2\} \cdot \{t_1, t_2\}$:
  - y_1, y_2 are x's sub-components
  - t_1, t_2 are x's transitions
- in transition $t_1 = \langle\{s_1, s_2\}, \{r\}, y\rangle$:
  - y is t_1's residue
  - s_1, s_2 are t_1's input signals
  - r is t_1's output signal

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The nature of composition

- Composition as algebraic operator:

$$op : \Sigma(X) \rightarrow X$$

- Composition and types:

$$\begin{tikzcd}
& c \models \gamma \arrow[dl, dash, "f_1"'] \arrow[dr, dash, "f_n"] & \\
a_1 \models \alpha_1 & \cdots & a_n \models \alpha_n
\end{tikzcd}$$

- Chu space $\mathbf{A} = (A, \Sigma, \models)$
    - A tokens (components), Σ types
    - $a \models \alpha$: token a is of type α

The nature of composition: effect on tokens

Composition identifies tokens in a composite:

$$\begin{tikzcd} & c \arrow[dl, "f_1^{\vee}"'] \arrow[dr, "f_n^{\vee}"] & \\ a_1 & \cdots & a_n \end{tikzcd}$$

The nature of composition: effect on types

Composition combines types in a composite:

$$
\begin{tikzcd}
 & \gamma & \\
\alpha_1 \arrow[ur, "f_1^{\wedge}"] & \cdots & \alpha_n \arrow[ul, "f_n^{\wedge}"']
\end{tikzcd}
$$

Composition as Chu-morphism

$$\begin{tikzcd}
\Sigma_A \arrow[r, "f^{\wedge}"] \arrow[d, dash, "\vDash_A"'] & \Sigma_C \arrow[d, dash, "\vDash_C"] \\
A & C \arrow[l, "f^{\vee}"]
\end{tikzcd}$$

$$c \vDash f_i^{\wedge}(\alpha_i) \iff f_i^{\vee}(c) \vDash \alpha_i$$

Systems as colimits of Chu spaces

Model an assemblage $\mathbf{A}$ as a collection of Chu spaces and Chu morphisms

$$\{\mathbf{A}_i\}_{i\in I} \;\cup\; \{f_k : \mathbf{A}_{i_k} \to \mathbf{A}_{j_k}\}_{k\in K}$$

Theorem

Every composition system has a colimit and it is unique up to isomorphism.

$$\begin{tikzcd} & \mathbf{C} & \\ \mathbf{A_i} \arrow[ur, "h_i"] \arrow[rr, "f"'] & & \mathbf{A_j} \arrow[ul, "h_j"'] \end{tikzcd}$$

Questions

- How to reconcile the two views ?
  - Components as pointed coalgebras
  - Composition as Chu-morphism
  - Cf Abramsky: category of Chu spaces as full subcategory of Grothendiek category of coalgebras
- Can we find a universal component functor ?
  - Every composition can be realized
- Can we find a universal language for the component functor ?
  - Every r.e. pointed coalgebra can be realized

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