The Geometry of Interaction Program
Proofs, Programs, Operators and Algorithmic Complexity

Thomas Seiller

Séminaire 68nqrt, IRISA, Rennes, June 13th 2013

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1. From proofs to graphs
2. Interaction Graphs
3. Logarithmic Space

From proofs to graphs

Linear Logic

Studying models of the λ-calculus, Girard realized that the implication $A \to B$ is decomposed as:

$$!A \multimap B$$

- $\multimap$ is a *linear* implication — which uses its argument exactly once;
- ! is a modality allowing the duplication;

He then introduced linear logic to mirror these semantics remarks in the syntax. The restriction to linear implication induces a splitting of the conjunction and disjunction into *multiplicative* ($\otimes$, $\parr$) and *additive* ($\&$, $\oplus$) variants.

Example

$$\dfrac{}{\vdash A, A^{\perp}}\ \mathrm{ax} \qquad \dfrac{\vdash \Gamma, A \qquad \vdash \Delta, A^{\perp}}{\vdash \Gamma, \Delta}\ \mathrm{cut}$$

$$\dfrac{\vdash \Gamma, A \qquad \vdash \Gamma, B}{\vdash \Gamma, \Delta, A \otimes B}\ \otimes \qquad \dfrac{\vdash \Gamma, A, B}{\vdash \Gamma, A \parr B}\ \parr$$

Example

$$\dfrac{A \vdash B \qquad \dfrac{\dfrac{}{A \vdash A}\,\text{ax} \qquad \dfrac{}{B \vdash B}\,\text{ax}}{A, B \vdash A \wedge B}\,\wedge}{\dfrac{\dfrac{A, A \vdash A \wedge B}{A \vdash A \wedge B}\,\text{ctr}}{\vdash A \to A \wedge B}\,\to}\,\text{cut}$$

Example

$$
\dfrac{\dfrac{A \vdash B \qquad \dfrac{\dfrac{}{A \vdash A}\,\mathrm{ax} \qquad \dfrac{}{B \vdash B}\,\mathrm{ax}}{A, B \vdash A \otimes B}\,\otimes}{\dfrac{A, A \vdash A \otimes B}{\color{red}{A \vdash A \otimes B}}\,\color{red}{\mathrm{ctr}}}\,\mathrm{cut}}{\color{red}{\vdash A \to A \otimes B}}\,\color{red}{\to}
$$

Example

$$
\dfrac{\dfrac{\dfrac{\dfrac{A \vdash B \qquad \dfrac{\dfrac{}{A \vdash A}\,\mathrm{ax} \qquad \dfrac{}{B \vdash B}\,\mathrm{ax}}{A, B \vdash A \otimes B}\,\otimes}{A, A \vdash A \otimes B}\,\mathrm{cut}}{!A, !A \vdash A \otimes B}\,\mathrm{der}}{\color{red}{A \vdash A \otimes B}}\,\mathrm{ctr}}{\color{red}{\vdash A \to A \otimes B}}\,\color{red}{\to}
$$

Example

$$
\begin{prooftree}
\AxiomC{$A \vdash B$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$A \vdash A$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$B \vdash B$}
\RightLabel{$\otimes$}
\BinaryInfC{$A, B \vdash A \otimes B$}
\RightLabel{cut}
\BinaryInfC{$A, A \vdash A \otimes B$}
\doubleLine
\RightLabel{der}
\UnaryInfC{$!A, !A \vdash A \otimes B$}
\RightLabel{ctr}
\UnaryInfC{$!A \vdash A \otimes B$}
\RightLabel{$\color{red}\rightarrow$}
\UnaryInfC{$\color{red}\vdash A \rightarrow A \otimes B$}
\end{prooftree}
$$

Example

$$
\dfrac{A \vdash B \qquad \dfrac{\dfrac{}{A \vdash A}\,\text{ax} \qquad \dfrac{}{B \vdash B}\,\text{ax}}{A, B \vdash A \otimes B}\,\otimes}{\dfrac{\dfrac{\dfrac{A, A \vdash A \otimes B}{!A, !A \vdash A \otimes B}\,\text{der}}{!A \vdash A \otimes B}\,\text{ctr}}{\vdash !A \multimap A \otimes B}\,\multimap}\,\text{cut}
$$

Restricting modalities

- Linear Logic allows a detailed control on duplication;
- Restricting the rules of the ! connective, one can define systems with less proofs which turn out to be interesting for the study of complexity.

For instance:
The ELL system (Elementary Linear Logic):

The proofs of type $!\mathrm{nat} \multimap \mathrm{nat}$ in ELL are exactly the functions computable in elementary time.

Restricting modalities

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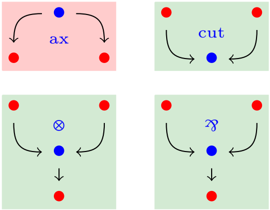
For instance:
The LLL system (Light Linear Logic):

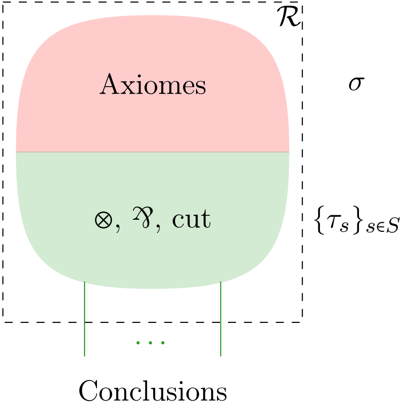
The proofs of type $!\mathrm{nat} \multimap \mathrm{nat}$ in LLL are exactly the functions computable in polynomial time.

Proof structures:



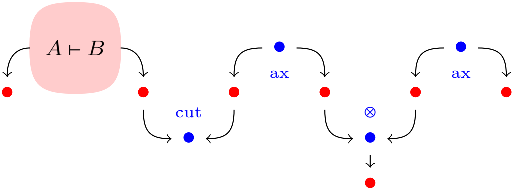
Théorème

$\mathcal{R}$ *is sequentializable*
iif
$\forall s \in S$, $\sigma\tau_s$ *is cyclic.*



Example

$$
\cfrac{A \vdash B \qquad \cfrac{\cfrac{}{A \vdash A}\,\text{ax} \qquad \cfrac{}{B \vdash B}\,\text{ax}}{A, B \vdash A \otimes B}\,\otimes}{\cfrac{A, A \vdash A \otimes B}{\cfrac{A \otimes A \vdash A \otimes B}{\vdash A \otimes A \multimap A \otimes B}\,\multimap}\,\otimes}\,\text{cut}
$$

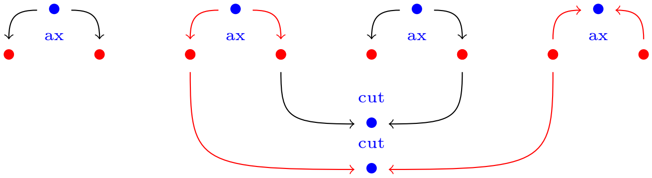


Geometry of Interaction: Interpretation of proofs

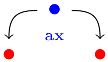
In a first approximation, the geometry of interaction program aims at defining a semantics of proofs that accounts for the dynamics of cut-elimination.

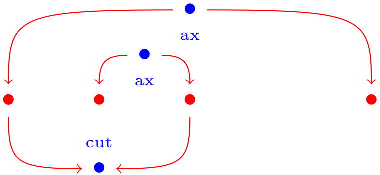
- A proof is interpreted by its set of axiom links;
- Cut-elimination corresponds to the computation of paths in the graph.

Example



Example





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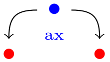

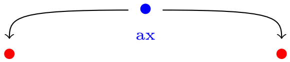

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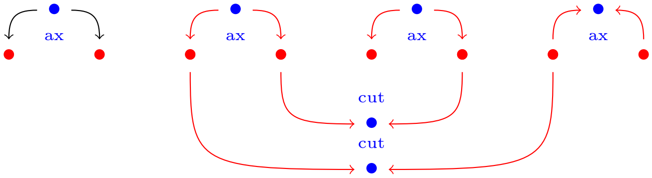


A remark on Proof Nets

- One can notice that the tests for A coming from the correctness criterion are in correspondence with the *proofs* of $A^\perp$.

$$\begin{array}{rcl} \text{Tests for A} & = & \text{Proofs of } A^\perp \\ \text{Proofs of A} & = & \text{Tests for } A^\perp \end{array}$$

Geometry of Interaction: Negation

Geometry of interaction aims at more than a mere interpretation of proofs. It is a complete reconstruction of logic around the dynamics of cut-elimination.

Based on the duality between tests for a A and proofs of $A^{\perp}$:

- one considers a set of more general objects (paraproofs) representing both proofs and tests;
- one defines a notion of orthogonality that translates the correctness criterion.

Geometry of Interaction: Formulas and Connectives

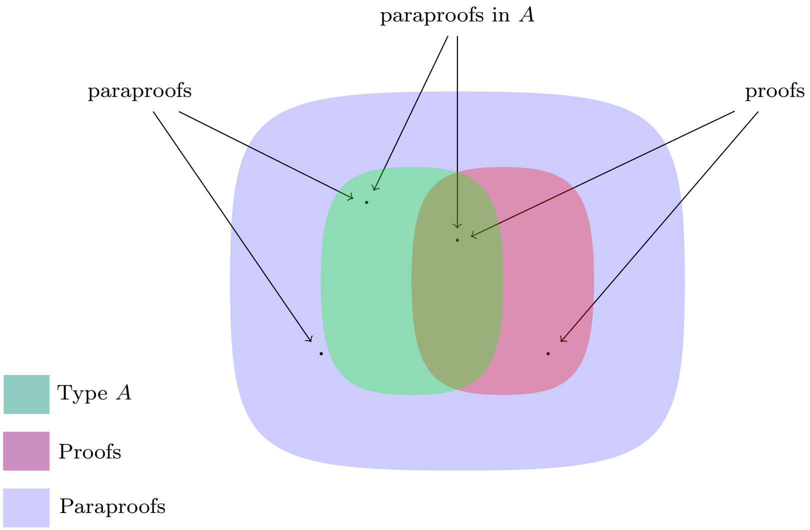
One defines formulas (or types) as sets of paraproofs closed by bi-orthogonality or, equivalently:

Definition

A formula is a set of paraproofs A such that $A = B^{\perp}$ for B a given set of tests.

Connectives are defined on paraproofs, and this definition induces a construction on types.

Principle


paraproofs in A
paraproofs
proofs
Type A
Proofs
Paraproofs

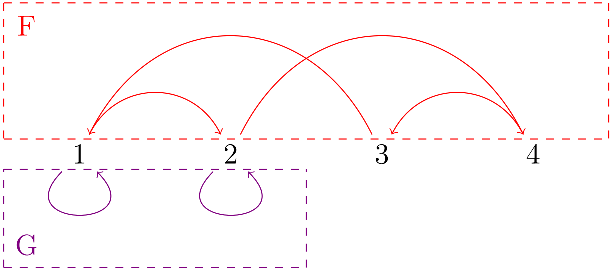
Interaction Graphs

Basics

- Paraproofs are graphs (with a natural number);
- Cut-elimination corresponds to taking the graph of paths;
- Orthogonality is defined by counting the cycles between two graphs.

Execution

The execution $F :: G$ of two graphs F, G is the graph of alternating paths of source and target in $V^F \Delta V^G$.



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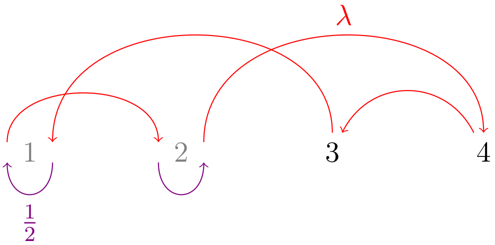

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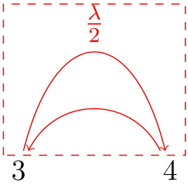
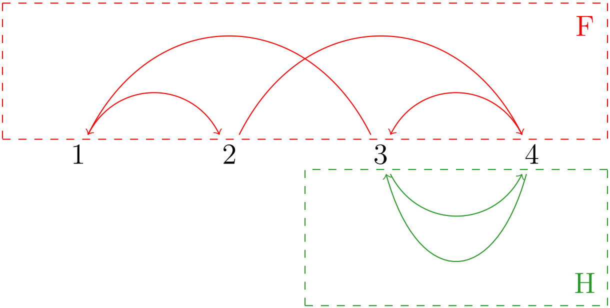


Cycles

In some cases, cycles appear during this operation.



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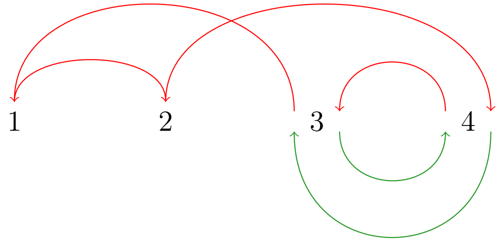

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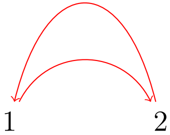

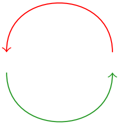

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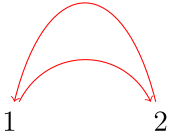
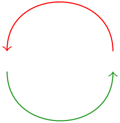




We will write $n_{F;G}$ the number of cycles appearing during the execution of the graphs F and G.

Counting Cycles

To take into account the appearance of cycles, we will count them in order to keep track of them.

Definition

A *paraproof* $\mathfrak{f}$ will be a graph F together with a natural number n_F.

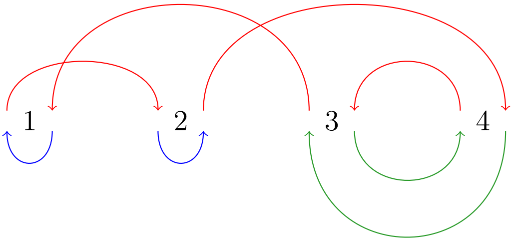
Definition

$$\ll\mathfrak{f},\mathfrak{g}\gg_m = n_F + n_G + n_{F;G}$$

Definition

The execution $\mathfrak{f} :: \mathfrak{g}$ of two paraproofs $\mathfrak{f} = (F, n_F)$ and $\mathfrak{g} = (G, n_G)$ is equal to $(F :: G, \ll\mathfrak{f},\mathfrak{g}\gg_m)$.
When F and G have no common vertices, $F :: G$ is the union of F and G, and $n_{F;G} = 0$. In this case, we will write $\mathfrak{f} \otimes \mathfrak{g}$ instead of $\mathfrak{f} :: \mathfrak{g}$.

The adjunction



Proposition

$$\langle\!\langle \mathfrak{f}, \mathfrak{g} \otimes \mathfrak{h} \rangle\!\rangle_m = \langle\!\langle \mathfrak{f} :: \mathfrak{g}, \mathfrak{h} \rangle\!\rangle_m$$

Definition

Two paraproofs $\mathfrak{f} = (F, n_F)$ and $\mathfrak{g} = (G, n_G)$ are orthogonal when they have the same sets of vertices and $\ll\mathfrak{f}, \mathfrak{g}\gg_m = 1$.

- This is the correctness criterion of proof nets.

Definition

A *type* is a set of paraproofs equal to its bi-orthogonal.

Definition

If $\mathbf{A}, \mathbf{B}$ are types, one can define:

$$\mathbf{A} \otimes \mathbf{B} = \{\mathfrak{a} \otimes \mathfrak{b} \mid \ \mathfrak{a} \in \mathbf{A}, \mathfrak{b} \in \mathbf{B}\}^{\perp\perp}$$

$$\mathbf{A} \multimap \mathbf{B} = \{\mathfrak{f} \mid \ \forall \mathfrak{a} \in \mathbf{A}, \mathfrak{f} :: \mathfrak{a} \in \mathbf{B}\}$$

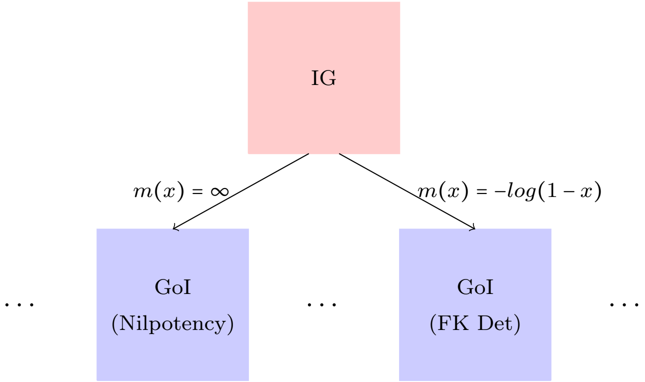
Proposition

$$\mathbf{A} \multimap \mathbf{B} = (\mathbf{A} \otimes \mathbf{B}^{\perp})^{\perp}$$

- We get a model of multiplicative linear logic.

Relation to Girard's constructions

Instead of counting cycles, we measure them (edges are weighted). We obtain a family of constructions parametrized by a "measure" m.



ELL

- In a generalization of this setting one can define several exponential connectives !.
- In particular, we can define a ! connective yielding a restricted system: ELL;

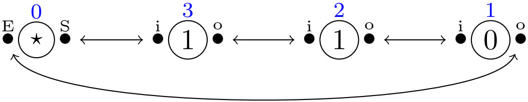
Logarithmic Space

joint work with Clément Aubert (Paris 13)

Representation of Integers

- Principle: an integer n is represented as a binary list, i.e. as a proof of
$$\underbrace{!(X \multimap X)}_{0} \multimap \underbrace{!(X \multimap X)}_{1} \multimap \underbrace{!(X \multimap X)}_{\star}$$
- The list can be read from the contraction rules.
- The GoI interpretation of these proofs are matrices (M_n) representing the sets of axiom links: we obtain a 6×6 matrix whose coefficients are $k \times k$ matrices ($k = \log_2(n)$ is the length of the list).

Representation of Integers: Example



3	$(0o,3)$	$(0i,3)$	$(1o,3)$	$(1i,3)$	$(S,3)$	$(E,3)$
2	$(0o,2)$	$(0i,2)$	$(1o,2)$	$(1i,2)$	$(S,2)$	$(E,2)$
1	$(0o,1)$	$(0i,1)$	$(1o,1)$	$(1i,1)$	$(S,1)$	$(E,1)$
0	$(0o,0)$	$(0i,0)$	$(1o,0)$	$(1i,0)$	$(S,0)$	$(E,0)$

$$\underbrace{!(X \multimap X)}_{0} \multimap \underbrace{!(X \multimap X)}_{1} \multimap \underbrace{!(X \multimap X)}_{\star}$$

- The blue indices in the previous slide are "states" of the integer (the integer is a function as in λ-calculus);
- As a consequence, a graph interacts with an integer only through the interface $\{0o, 0i, 1o, 1i, S, E\}$;
- We say a graph G (which can have "states" in the same way integers do) accepts an integer n when there are no cycles between G and M_n;
- The language $[G]$ recognized by a graph G is the set of integers it accepts.

Results

- In the formal setting:

$$\underbrace{\mathcal{M}_6(\mathbb{C}) \otimes \mathcal{R}}_{\text{integers}} \otimes \mathcal{M}_n(\mathbb{C})$$

  we define "machines" as a set R of operators in $\mathcal{M}_6 \otimes \mathcal{M}_n$ and we can show that $[R] = \{[r], r \in R\}$ is the set of regular languages.
- In the more complex setting:

$$\mathcal{M}_6(\mathbb{C}) \otimes \left((\bigotimes_{n \in \mathbb{N}} \mathcal{R}) \rtimes \mathfrak{G} \right) \otimes \mathcal{M}_n(\mathbb{C})$$

  we define two sets P_+ and $P_{+,1}$ of operators in $\mathcal{M}_6 \otimes \mathcal{M}_n$ and show:

$$[P_+] = \textbf{co-NL} \qquad [P_{+,1}] = \textbf{L}$$

Conclusion

Perspectives

- Use tools and/or invariants of operator algebras to obtain results in algorithmic complexity;
- Use the concepts of GoI to study other notions of computation:
  - Quantum computation;
  - Pi-calculus;
  - ...
- Understand the links with homotopy.