The probabilistic modal μ-calculus with independent product

Matteo Mio
University of Edinburgh, School of Informatics, LFCS

The Modal μ-Calculus: Lμ

- Introduced by D. Kozen (1983).
- Interpreted on LTS$= \langle P, \{\xrightarrow{a}\}_{a \in L}\rangle$, with $\xrightarrow{a} \subseteq P \times P$.
- Syntax: $F ::= X \mid F \vee G \mid F \wedge G \mid \langle a \rangle F \mid [a] F \mid \mu X.F \mid \nu X.F$
- Semantics: $[\![F]\!]_\rho \subseteq P$, with $\rho : Var \to 2^P$

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\begin{aligned}
[\![F]\!]_\rho &= \rho(X) \\
[\![F \vee G]\!]_\rho &= [\![F]\!]_\rho \cup [\![G]\!]_\rho \\
[\![F \wedge G]\!]_\rho &= [\![F]\!]_\rho \cap [\![G]\!]_\rho \\
[\![\langle a \rangle F]\!]_\rho &= \{p \mid \exists q. p \xrightarrow{a} q \ , \ q \in [\![F]\!]_\rho\} \\
[\![[a] F]\!]_\rho &= \{p \mid \forall q. p \xrightarrow{a} q \text{ implies } q \in [\![F]\!]_\rho\} \\
[\![\mu X.F]\!]_\rho &= \text{lfp of } \lambda S. [\![F]\!]_{\rho[S/X]} \\
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Expressivity: bisimilarity-invariant fragment of *MSO* (Janin, Walukiewicz 1996).

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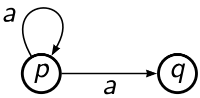
Game Semantics: a few examples



$$F = \nu X.\langle a \rangle X$$

There exist a infinite *a*-path.

$$G = \mu X.[a]X$$

Every *a*-path is finite.

$$H = \langle a \rangle [a] \bot$$

There is some *a*-step, after which no further *a*-step is possible.

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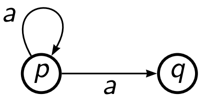

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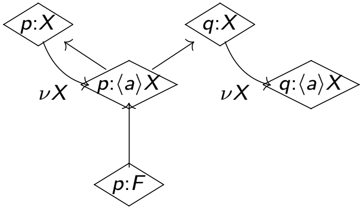
Game associated with F:



Player $\Diamond$ either gets stuck, or can force the play into an infinite ν-loop.

So, $\Diamond$ wins this game: $[\![F]\!](p) = 1$.

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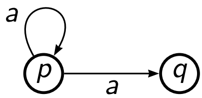

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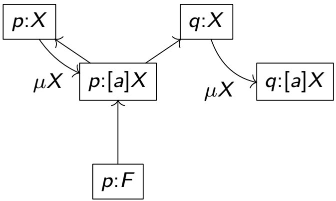
Game associated with G:



Player $\Box$ either gets stuck, or can force the play into an infinite μ-loop.

So, $\Box$ wins this game, i.e. $\Diamond$ loses: $[\![G]\!]\,(p) = 0$.

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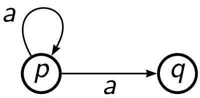

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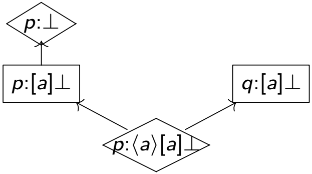

Player $\Diamond$ can make sure
Player $\Box$ will get stuck.

So, $\Diamond$ wins this game:
$[\![H]\!]\,(p) = 1$.

Theorem [e.g. Stirling 96]:
$[\![F]\!](p) = 1$ iff $\Diamond$ has a winning strategy in $\mathcal{G}^F$ from $(p : F)$.

- Denotational Semantics and Game Semantics coincide.
- Useful to have an operational interpretation for the meaning of a formula.
- Game Semantics very successful: theoretical results, model checking algorithms, ...

Probabilistic LTS and Probabilstic μ-calculus

A PLTS is a pair $\langle P, \{\xrightarrow{a}\}_{a \in L} \rangle$ where

- P is a countable set of states,
- L is a countable set of labels, or *atomic* actions,
- $\xrightarrow{a} \subseteq P \times \mathcal{D}(P)$ is the a-transition relation.

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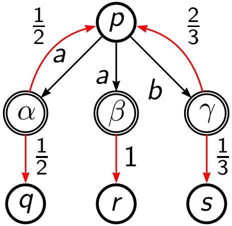

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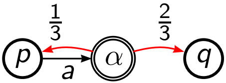
$$[\![F]\!](\alpha) = \sum_{p \in P} \alpha(p) \cdot [\![F]\!](p)$$

Game Semantics for pLμ: a few examples



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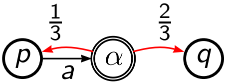

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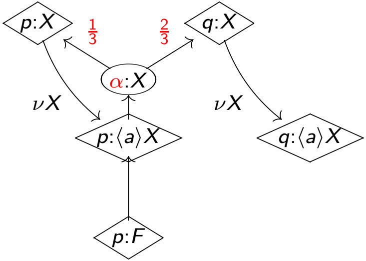

Player $\Diamond$ either gets stuck (and lose), or end up in a infinite ν-loop (and win). **However**, the probability of winning is 0.

So, $\Diamond$ wins this game with probability 0: $[\![F]\!](p) = 0$.

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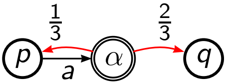

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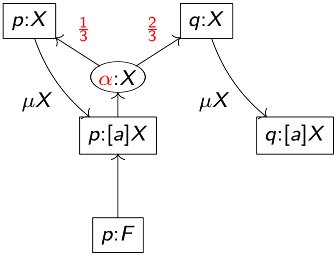
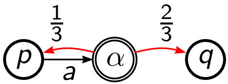
Game associated with G:



Player $\Box$ either gets stuck (and lose), or end up in a infinite μ-loop (and win). **However**, this happens with prob. 0.

So, $\Diamond$ wins this game with probability 1: $[\![G]\!]\,(p) = 1$.

Game Semantics for pLμ: a few examples



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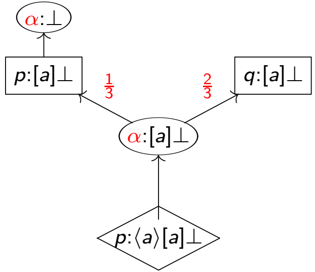
Game associated with H:



Player $\Diamond$ reaches $\bot$ with prob. $\frac{1}{3}$, and $\Box$ gets stuck with probability $\frac{2}{3}$.

So, $\Diamond$ wins this game with probability $\frac{2}{3}$: $[\![H]\!]\,(p) = \frac{2}{3}$.

$L\mu$	$pL\mu$
$F = \nu X.\langle a \rangle X$ There exist a infinite *a*-path.	$F = \nu X.\langle a \rangle X$ Best probability of producing an infinite *a*-path.
$G = \mu X.[a] X$ Every *a*-path is finite.	$G = \mu X.[a] X$ Probability that any adversary environment, fails in producing an infinite *a*-path
$H = \langle a \rangle [a] \bot$ There is some *a*-step, after which no further *a*-step is possible.	$H = \langle a \rangle [a] \bot$ Probability to reach after some *a*-step a state without *a*-edges.
In general	Best probability of satisfying *F* (read as in $L\mu$) against any hostile environment.

Theorem [Mio 2010, Morgan and McIver 2004]:

$$[\![F]\!](p) = \text{value of the game } \mathcal{G}^F \text{ at } (p : F).$$

where the (quantitative) value is defined as usual in game theory:

$$\bigsqcup_{\sigma_\Diamond} \bigsqcap_{\sigma_\Box} E(M_{\sigma_\Diamond,\sigma_\Box}) = \bigsqcap_{\sigma_\Box} \bigsqcup_{\sigma_\Diamond} E(M_{\sigma_\Diamond,\sigma_\Box})$$

- Denotational Semantics and Game Semantics coincide.
- Useful to have an operational interpretation for the meaning of a probabilistic sentences.
- Game Semantics provides: model checking algorithms, ...

Prob. μ-Calculus with Independent Product: pL$\mu^{\odot}$

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- Syntax: $F ::= X \mid F \vee G \mid F \wedge G \mid \langle a \rangle F \mid [a] F \mid \mu X.F \mid \nu X.F$
$$F \odot G \mid F \cdot G$$
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$$[\![F \cdot G]\!](p) = [\![F]\!](p) \cdot [\![G]\!](p) \quad [\![F \odot G]\!](p) = [\![F]\!](p) \odot [\![G]\!](p)$$
- where $x \odot y = x + y - (x \cdot y)$
  - the De Morgan dual of $\cdot$ under $\neg x = 1 - x$: $x \odot y \stackrel{\text{def}}{=} \neg(\neg x \cdot \neg y)$.

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- Mathematically well defined ($\cdot$ and $\odot$ are monotone).
- But is it meaningful?
  - $[\![F \cdot G]\!]$ probability that F **and** G holds independently?
  - $[\![F \odot G]\!]$ probability that F **or** G holds independently?

Why pL$\mu^{\odot}$??

Let us define

- $\mathbb{P}_{>0}F \stackrel{\text{def}}{=} \mu X.(F \odot X)$, and
- $\mathbb{P}_{=1}F \stackrel{\text{def}}{=} \nu X.(F \cdot X)$.

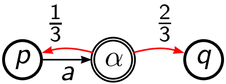
Then

- $[\![\mathbb{P}_{>0}F]\!](p) = \begin{cases} 1 & \text{if } [\![F]\!](p) > 0 \\ 0 & \text{otherwise} \end{cases}$
- $[\![\mathbb{P}_{=1}F]\!](p) = \begin{cases} 1 & \text{if } [\![F]\!](p) = 1 \\ 0 & \text{otherwise} \end{cases}$

This allows:

- the expression of interesting (new) properties involving qualitative/quantitative assertions (see paper).
- The encoding of the qualitative fragment of PCTL into pL$\mu^{\odot}$.

Game Semantics for pLμ: a few examples



$$H = \langle a \rangle [a] \bot$$

Probability to reach after some a-step a state without a-edges.

$$J = \langle a \rangle \langle a \rangle \top$$

Probability to reach after some a-step a state with some a-edge.

$$H \cdot J$$

Probability of satisfying both H **and** J when H and J independently verified.

Game associated with $H \cdot J$:

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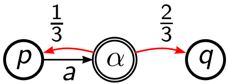

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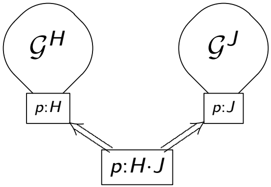

At the state $p : H \cdot J$ the game is split in two concurrent and independent sub-games.

$\Diamond$ wins iff He wins in **both** sub-games.

Since they are independent, this will happen with probability $\frac{1}{3} \cdot \frac{2}{3}$.

$[\![H \cdot J]\!](p) = \frac{2}{9}$.

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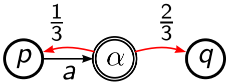

$$H = \langle a \rangle \, [a] \perp$$

Probability to reach after some a-step a state without a-edges.

$$\mathbb{P}_{>0} H = \mu X.(X \odot H)$$

1 if H is possible,

0 otherwise.

$$\Updownarrow$$

probability that H holds at least once if verified infinitely many times.

Game associated with $\mu X.(H \odot X)$:

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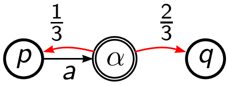

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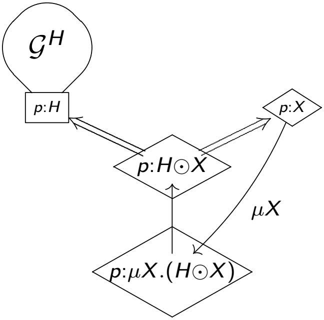
$\Updownarrow$

probability that H holds at least once if verified infinitely many times.

Game associated with $\mu X.(H \odot X)$:



$\Diamond$ will win in at least on sub-game almost surely!

$$[\![\mu X.(H \odot X)]\!](p) = 1.$$

- These ideas are formalized using $2\frac{1}{2}$-player tree games, which build on the intuitive idea of concurrent and independent execution of sub-games.
- A new class of games having trees as outcomes, rather than paths.
  - The branches of the trees are generated when the game is split in concurrent and independent sub-games.
- The winning-set of a $2\frac{1}{2}$-player tree game is a set Φ of trees
  - which we call **branching plays**.
- In the case of pL$\mu^{\odot}$ games the winning set, is the set of trees
  - such that $\Diamond$ can find a winning path by making **or** choices at the branching nodes $p : F \odot G$, against any **and** choice made by $\Box$ on the nodes $p : F \cdot G$.
  - i.e. the trees that, once interpreted as ordinary 2-player parity games, are won by $\Diamond$.
- That's why we call them $2\frac{1}{2}$-player meta -parity games.

- One can define the notion of (upper and lower) **value** of a $2\frac{1}{2}$-player tree game.
- $Val_{\downarrow}(\mathcal{G}) = \bigsqcup_{\sigma_{\Diamond}} \bigsqcap_{\sigma_{\Box}} E_{\sigma_{\Diamond},\sigma_{\Box}}(\Phi)$
- $Val_{\uparrow}(\mathcal{G}) = \bigsqcap_{\sigma_{\Box}} \bigsqcup_{\sigma_{\Diamond}} E_{\sigma_{\Diamond},\sigma_{\Box}}(\Phi)$

Theorem ($\mathbf{MA}_{\aleph_1}$): If $\mathcal{G}$ is a pL$\mu^{\odot}$ game, then:

$$Val_{\downarrow}(\mathcal{G}) = Val_{\uparrow}(\mathcal{G}).$$

Theorem ($\mathbf{MA}_{\aleph_1}$): For every pL$\mu^{\odot}$ formula F:

$$[\![F]\!](p) = \text{value of } \mathcal{G}^F \text{ at } (p, F).$$

Our theorems hold in $\mathrm{ZFC}+\mathrm{MA}_{\aleph_1}$ set theory.

- MA is an axiom considered by set theorists as a weaker alternative to CH.
- $\mathrm{MA}_{\aleph_1}$ is a consequence of $\mathrm{MA}+\neg\mathrm{CH}$ and itself implies $\neg\mathrm{CH}$.
- In particular it implies that:
  - measurable sets are closer under ω_1 unions.
  - measures are ω_1-continuous.
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- In particular it implies that:
  - measurable sets are closer under ω_1 unions.
  - measures are ω_1-continuous.
- Therefore our proof is a *consistent proof*.
  - Also Fermat's Last Theorem is proved in ZFC + U!! $\forall a, b, c \in \mathbb{Z}. a^n + b^n \neq c^n$, when $n > 3$.

We use $\mathrm{MA}_{\aleph_1}$ to handle the complexity of the winning sets Φ of pL$\mu^{\odot}$ games.

- We prove that Φ is always a $\mathbf{\Delta}^1_2$ set.
- Hence not Borel, and not necessarily measurable.
- But we characterize Φ as a ω_1-union of measurable sets: $\Phi = \bigcup_{\alpha<\omega_1} \Phi^{\alpha}$.
- Hence, under $\mathrm{MA}_{\aleph_1}$, Φ is measurable, and its measure is the limit of the measures of the approximants.
  - $\mu(\Phi) = \bigsqcup_{\alpha<\omega_1} \mu(\Phi^{\alpha})$

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  - Failure of positional determinacy makes this problem challenging.
- Study the logical-equivalence (or metric) induced by the logic pL$\mu^{\odot}$, or even the modal fragment $\{\top, \bot, \vee, \wedge, \langle a \rangle, [a], \cdot, \odot\}$.

THANKS

A few interesting examples

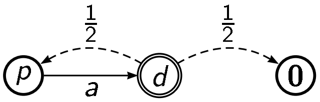

Figure: Example of PLTS

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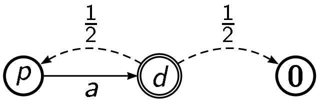

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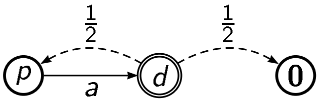

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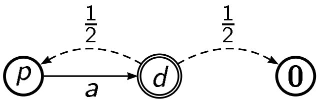
3. $F_5 \stackrel{\text{def}}{=} \langle a \rangle \langle a \rangle \underline{1} \wedge [a]\,[a]\,\underline{0}$ $\qquad 0 \leq [\![F_5]\!]\,(p) \leq \frac{1}{2}$ for all p
   The logic is not Boolean! (Kleene Algebra)



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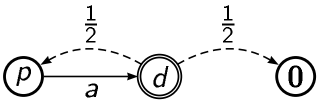

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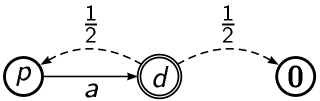

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3. $G_5 \stackrel{\text{def}}{=} \mathbb{P}_{>0}\Big(\mu X.\big(G_1 \vee \langle b\rangle X\big)\Big)$
   "Holds iff the probability (above) is greater than 0".

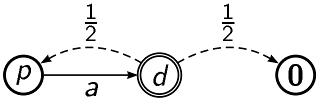

Figure: Example of PLTS

1. $H_1 = \nu X.\mathbb{P}_{>0}\langle a \rangle X$
   "Holds if it is possible to make infinitely many **possible** a's:
   $p \xrightarrow{a} d_1 \rightsquigarrow p_1 \xrightarrow{a} d_2 \rightsquigarrow p_2 \ldots$ with $d_n(p_n) > 0$
2. $H_2 = \mu X.\mathbb{P}_{=1}[a]X$
   Dual of H_1: "holds if it is impossible to make infinitely many **possible** a's:
3. $H_3 = \mu X.\big((\mathbb{P}_{>0}\langle a \rangle X) \vee \mathbb{P}_{=1} H\big)$
   "Holds if it is possible to make finitely many **possible** a's and reach a state where H holds with probability 1.