Towards a general construction of playgrounds

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Outline

Introduction

- notion of playground

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- instances with similar constructions

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- general and automated construction of playgrounds

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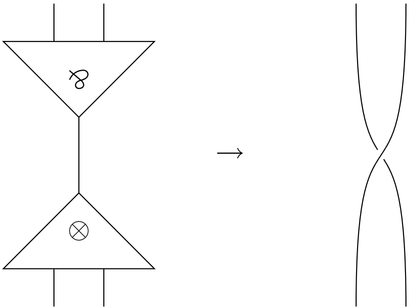
- notion of playground
- instances with similar constructions
- general and automated construction of playgrounds
- simple playground to demonstrate the categorical combinatorics methods

An MLL interaction net

Reduction in MLL interaction nets



Let us take a closer look at the interaction net

The base category

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- objects of dimension 0 are called *channels*
- objects of dimension 1 are called *players*
- objects of dimension at least 2 are called *moves*

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In the case of MLL interaction nets:

$*$ dimension 0

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$\otimes$	$\parr$	dimension 1
$*$		dimension 0

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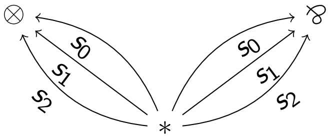

dimension 1

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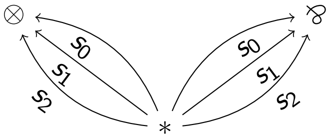
In the case of MLL interaction nets:

cut — dimension 2



dimension 1

dimension 0

The base category

In the case of MLL interaction nets:

$$
\begin{tikzcd}
 & \mathrm{cut} & & & \text{dimension } 2 \\
\otimes \arrow[ur, "l"] & & \parr \arrow[ul, "r"'] & & \text{dimension } 1 \\
 & * \arrow[ul, bend right=40, "s_0"'] \arrow[ul, "s_1"] \arrow[ul, bend left=40, "s_2"] \arrow[ur, bend left=40, "s_0"] \arrow[ur, "s_1"'] \arrow[ur, bend right=40, "s_2"'] & & & \text{dimension } 0
\end{tikzcd}
$$

with the relations $l \circ s_i = r \circ s_i$.

Positions as presheaves

A presheaf on a category $\mathcal{C}$ is a functor $F : \mathcal{C}^{op} \longrightarrow \mathrm{Set}$.
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Positions are a "suitable" subcategory of presheaves of dimension 1 on $\mathcal{C}$.
In the case of MLL interaction nets, positions are presheaves of dimension 1 such that there are never more than two morphisms from players to a given channel.

Positions as presheaves

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- objects: pairs (c, x) where c is an object of $\mathcal{C}$ and $x \in F(c)$
- morphisms: $u : (c, x) \to (c', x')$ if $u : c \to c'$ is a morphism of $\mathcal{C}$ such that $F(u)(x') = x$.

Category of elements

Seeds

We assume given, for every move m of $\mathcal{C}$, a cospan in the category of presheaves $Y \rightarrow M \leftarrow X$ (called a "seed"), where X and Y are positions, M is the representable presheaf corresponding to m and that verifies the following properties:

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- $X(p) + Y(p) \to M(p)$ is surjective for every player p
- it has a "canonical interface" I of dimension 0 such that:

$$\begin{tikzcd} & I \arrow[dl] \arrow[d] \arrow[dr] & \\ X \arrow[r] & M & Y \arrow[l] \end{tikzcd}$$

commutes.

Seeds

Moves

A move is a cospan $Y \rightarrow M \leftarrow X$ built from a seed $Y_0 \rightarrow M_0 \leftarrow X_0$ with canonical interface I by pushing out along some "suitable" position Z in the following way:

$$
\begin{tikzcd}
 & Y_0 \arrow[rr] \arrow[d] & & Y \arrow[d, dotted] \\
 & M_0 \arrow[rr] & & M \\
I_0 \arrow[uur] \arrow[ur] \arrow[rr] \arrow[dr] & & Z \arrow[uur] \arrow[ur] \arrow[dr] & \\
 & X_0 \arrow[uu] \arrow[rr] & & X \arrow[uu, dotted]
\end{tikzcd}
$$

Moves

Building plays

A play is a composition of moves in the bicategory of cospans.

$$
\begin{tikzcd}
X_n \arrow[dr] & & \\
& M_n \arrow[ddr] & \\
\vdots \arrow[ur] \arrow[dr] & \vdots & \\
& M_2 \arrow[r] & U \\
X_1 \arrow[ur] \arrow[dr] & & \\
& M_1 \arrow[uur] & \\
X_0 \arrow[ur] & &
\end{tikzcd}
$$

A first example

A second example

Problem

- plays are cospans $Y \to U \leftarrow X$, but when is such a cospan a play?

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- plays are cospans $Y \to U \leftarrow X$, but when is such a cospan a play?
- need for a correctness criterion.

Past

For every seed $Y \overset{s}{\hookrightarrow} M \overset{t}{\hookleftarrow} X$, we define the past of M to be the set:

$$\mathrm{past}(M) = \bigcup_{p \in \mathcal{C}} M(p) - Y(p)$$

Cores, core separation

A core of a presheaf U is a move μ of $\mathrm{el}(U)$ such that, if $f : \mu \to x$ in $\mathrm{el}(U)$, then $x = \mu$ and $f = id_\mu$.

A presheaf $U \in \widehat{\mathcal{C}}^f$ is core-separating if for all cores $\mu \neq \mu'$ in $\mathrm{el}(U)$, the pullback of μ along μ' is a position.

Local 1-injectivity

A presheaf U on $\mathcal{C}$ is locally 1-injective iff for every seed $Y \overset{s}{\hookrightarrow} M \overset{t}{\hookleftarrow} X$ with canonical interface $u : I \to M$ and for all corresponding core $\mu \in \mathrm{el}(U)$ (seen as a morphism $\mu : M \to U$), if $x \neq y \in M$ are such that $\mu(x) = \mu(y)$, then x, y are in the image of u.

Partitioning players and channels

For every seed, we partition players in the following way:

- consumed players: $\mathrm{Co}(M)(p) = X(p) \setminus Y(p)$
- created players: $\mathrm{Cr}(M)(p) = Y(p) \setminus X(p)$
- surviving players: $\mathrm{Sr}(M)(p) = X(p) \cap Y(p)$

We do the same for channels.

Initial and final players and channels

For every presheaf $U \in \widehat{\mathcal{C}}^f$, we define the set of its initial and final players by:

- $\mathrm{Init}(U)(p) = \{x \in U(p) \mid \nexists m \in \mathcal{C},\ \widetilde{m} \in U(m),\ x \in \mathrm{Cr}(\widetilde{m})(p)\}$
- $\mathrm{Fin}(U)(p) = \{x \in U(p) \mid \nexists m \in \mathcal{C},\ \widetilde{m} \in U(m),\ x \in \mathrm{Co}(\widetilde{m})(p)\}$

The causal graph

We define the causal graph G_U that has:

G_U is source-linear if for all $x \to \mu$, $x \to \mu'$, $\mu = \mu'$.

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  - $x \to x \cdot s$ for every player x and $s : p \to *$

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  - $\mu \longrightarrow x$ for every core μ and $x \in \mathrm{Co}(\mu)(*) \cup \mathrm{Sr}(\mu)(*)$
  - $\mu \longrightarrow \mu'$ for all cores $\mu \neq \mu'$ when there is a player x in $\mathrm{Co}(\mu) \cap (\mathrm{Co}(\mu') \cup \mathrm{Sr}(\mu'))$

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Assumed

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- for every seed M, $\mathrm{past}(M)$ only contains moves and players
- there is no isolated channel
- every seed $Y \xhookrightarrow{s} M \xhookleftarrow{t} X$ has a canonical interface $I = X_0$ ($X_0(x) = X(x)$ in dimension 0, $X_0(x) = \varnothing$ otherwise)

The correctness criterion

Correctness Criterion

A cospan $Y \hookrightarrow U \hookleftarrow X$ is a play iff the following conditions are met:

- U is core-separating and locally 1-injective
- X contains exactly the initial players and channels of U
- Y contains exactly the final players and channels of U
- G_U is source-linear and acyclic

Lemma

Lemma

Let U be a presheaf on $\mathcal{C}$ and μ be maximal in G_U (i.e., there is no path from μ to any other core). Assume that U is core-separating. Then for all $c \in \mathrm{el}(U)$, $U(c) - \mathrm{past}(\mu) = (U \setminus\!\setminus \mu)(c)$.

Sketch of the proof

Take μ_0 maximal in G_U, and be

$$
\begin{tikzcd}
 & I_0 \arrow[dl, "h_0"'] \arrow[d, "l_0"] \arrow[dr, "k_0"] & \\
X_0 \arrow[r, "t_0"'] & M_0 & Y_0 \arrow[l, "s_0"]
\end{tikzcd}
$$

its canonical interface.

Sketch of the proof

$$\begin{tikzcd}
 & & & & & Y \arrow[dl, dotted] \arrow[ddd, "s"] \\
 & & & & U' \arrow[ddr] & \\
 & Y_0 \arrow[rr] \arrow[d] & & Z \arrow[ur, dotted] \arrow[d, dotted, "s'"] & & \\
 & M_0 \arrow[rr] \arrow[rrrr, bend left, "\mu_0"] & & M \arrow[rr, dotted, "\mu"'] & & U \\
 I_0 \arrow[uur] \arrow[ur] \arrow[rr] \arrow[dr] & & X_1 \arrow[uur] \arrow[ur] \arrow[dr] & & & \\
 & X_0 \arrow[uu] \arrow[rr, "h"'] & & X \arrow[u, dotted, "t'"] \arrow[uurr, "t"'] & &
\end{tikzcd}$$

Conclusion

- MLL plays as cospans of presheaves

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